Zeros and Logarithmic Asymptotics of Sobolev Orthogonal Polynomials for Exponential Weights

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Abstract

We obtain the (contracted) weak zero asymptotics for orthogonal polynomials with respect to Sobolev inner products with exponential weights in the real semiaxis, of the form $x^\gamma e^{-\varphi(x)}$, with $\gamma > 0$, which include as particular cases the counterparts of the so-called Freud (i.e., when φ has a polynomial growth at infinity) and Erdös (when φ grows faster than any polynomial at infinity) weights. In addition, the boundness of the distance of the zeros of these Sobolev orthogonal polynomials to the convex hull of the support and, as a consequence, a result on logarithmic asymptotics is derived.

1 Introduction and main results

In this paper, the location and the asymptotic behavior of zeros of polynomials orthogonal with respect to a Sobolev inner product with exponential weights in the semiaxis is analysed. As a consequence, we obtain the logarithmic (n-root) asymptotics for the Sobolev orthogonal polynomials for exponential weights. To this end, a result about asymptotically extremal polynomials with respect to varying weights, established by the authors in a previous paper (see [1, Theorem 1.1]), plays a crucial role. Along with it, we give a new, and simpler, proof of the boundness of the distance of the zeros of these Sobolev orthogonal polynomials to the convex hull of the support, previously obtained by Durán and Saff [2].

Polynomials orthogonal with respect to a Sobolev inner product have been object of an extensive study during the last fifteen-twenty years. Indeed, let $\{\mu_i\}_{i=0}^M$ be a set of $M+1$ positive Borel measures supported on $\Delta \subset \mathbb{R}$. On the linear space $\mathbb{P}$ of all polynomials we introduce the inner product

$$\langle p, q\rangle_S := \sum_{i=0}^{M} \int_\Delta p^{(i)}(x)\, \overline{q}^{(i)}(x)\, d\mu_i(x)\,, \tag{1}$$

and the corresponding Sobolev norm of a polynomial p is given by:

$$\|p\|_S^2 \;=\; \sum_{i=0}^{M} \|p^{(i)}\|^2_{L^2(d\mu_i)}\,. \tag{2}$$

When Δ is a compact subset of the complex plane, we have a lot of information about the algebraic properties and analytic/asymptotic behavior of the system of orthogonal polynomials with respect to the above inner product (see for instance the survey [10]). On the contrary, the case where Δ is unbounded has only been considered recently. As it is well known, one of the main facts in the theory of standard orthogonality (i.e., when $M = 0$) is that the zeros of the orthogonal polynomials lie in the convex hull of the support of the measure μ_0; but in the Sobolev case ($M > 0$) this is not longer true. In this sense, in [2] it is obtained an upper bound for the distance of the zeros to the convex hull of the support, under certain conditions on the measures. For an updated review on the analytic properties of Sobolev polynomials orthogonal with respect to exponential type weights supported on unbounded sets of the real line we recommend [9] and for a more condensed reviewing the introduction of [8] and [1]. The results obtained here are the analogues of situations studied in [8] and [1], for the case where the support is the real semiaxis, with the natural aided difficulties. See also [3].

In the sequel, we consider a Sobolev inner product (1), with $\Delta = \mathbb{R}^+ = [0, \infty)$ and weights ρ_k^2, where:

$$\rho_k(x) \;=\; x^{\gamma} \exp(-\varphi_k(x))\,,\; \gamma > 0 \tag{3}$$

and

$$\varphi_k(x) = \exp_{\eta_k}(x^{\alpha_k}) - \exp_{\eta_k}(0)\,,\; \eta_k \in \mathbf{Z}_+\,,\; \alpha_k > \begin{cases} 0\,, & \text{if } \eta_k > 0\,, \\ 1\,, & \text{if } \eta_k = 0\,. \end{cases} \tag{4}$$

Here, $\exp_l(x)$ denotes the *lth iterated exponential*, defined as

$$\exp_l(x) = \begin{cases} \underbrace{\exp(\exp(\exp(\ldots\exp(x)))}_{l \text{ times}} & \text{if} \quad l > 0\,, \\ \qquad\qquad x & \text{if} \quad l = 0\,. \end{cases} \tag{5}$$

In [6]-[7] several properties of (standard) orthogonal polynomials with respect to such exponential weights in the real semiaxis are analyzed. See also [5] and [11, sect. 6].

In [1] a similar analysis is carried out, but for Sobolev inner products with exponential weights supported in the whole real axis. In order to extend those results to the case where the support is the semiaxis it is possible to follow a similar approach, but some modifications are needed.

For exponential weights ρ_k supported in the whole real axis it is well known that there exists a constant $a_{k,n}$, usually called the Mhaskar-Rakhmanov-Saff number, such that the sup norm of weighted polynomials $P\rho_k$, with $P \in \mathbb{P}_n$, essentially lives in the bounded interval $[-a_{k,n}, a_{k,n}]$ (see [12] or [5]). In the case of exponential weights of type (3), supported in the semiaxis:

$$\|P\rho_k\|_{L^\infty[0,\infty)} \;=\; \|P\rho_k\|_{L^\infty[0,a_{k,n}]}\,, \tag{6}$$

for any polynomial P of degree $\leq n$. The Mhaskar-Rakhmanov-Saff number $a_{k,n}$ is given, in the "Freud" case ($\eta_k = 0$), by (see [6, Theorem 5.1]):

$$a_{k,n} = (\gamma_k n)^{1/\alpha_k}\;,\; \gamma_k \;=\; \frac{\Gamma(\alpha_k)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\alpha_k + \frac{1}{2}\right)}\,. \tag{7}$$

For the "Erdös" case ($\eta_k > 0$), we have the following estimation of the MRS number (see [6, Theorem 5.1]):

$$a_{k,n} \;\sim\; \left(\log_{\eta_k}(n)\right)^{(1/\alpha_k)}\,, \tag{8}$$

where $\log_l$ denotes the l-th iterated logarithm (defined as the iterated exponential in (5)), and the expression "$c_n \sim d_n$" means that there exist two positive constants A, B such that $Ad_n \leq c_n \leq Bd_n$.

On the sequel, it will be convenient to consider the main weight, that is, the weight which dominates the other ones in the sense of the behavior at infinity. Indeed, we have that there exist $\overline{k} \in \{0, 1, \ldots, M\}$ and a constant $C > 0$ such that for each $0 \leq k \leq M$ the following inequality holds

$$\rho_k(x) \leq C\rho_{\overline{k}}(x)\,,\; x \in \mathbb{R}^+\,, \tag{9}$$

with $\quad \overline{k} := \min_{\eta_k=\overline{\eta},\alpha_k=\overline{\alpha},\gamma_k=\overline{\gamma}} k$, where $\overline{\eta} := \min_{0\leq k\leq M} \eta_k$, $\overline{\alpha} := \min_{\{k:\,\eta_k=\overline{\eta}\}} \alpha_k$, $\overline{\gamma} := \max_{\{k:\,\eta_k=\overline{\eta},\alpha_k=\overline{\alpha}\}} \gamma_k$.

Hereafter we shall refer to $\rho_{\overline{k}}(x) = x^{\overline{\gamma}} \exp(-\varphi_{\overline{k}}(x))$ as the main weight.

In order to show the results on asymptotic behavior of these Sobolev orthogonal polynomials, some definitions and results in potential theory will be recalled. First, remind the notion of admissible weights (see [12, Def.I.1.1]).

Definition 1.1 *Given a closed set $\Sigma \subset \mathbb{C}$, we say that a function $\omega : \Sigma \longrightarrow [0, \infty)$ is an admissible weight on Σ if the following conditions are satisfied:*

(i) ω is upper semi-continuous;

(ii) the set $\{z \in \Sigma : \omega(z) > 0\}$ has positive (logarithmic) capacity;

(iii) if Σ is unbounded, then $\lim_{|z|\to\infty,\, z\in\Sigma} |z|\omega(z) = 0$.

Given such an admissible weight ω in the closed set Σ, and setting $\phi(z) = -\log \omega(z)$, we know (see e.g. [12, Ch.I]) that there exists a unique measure μ_ω, with (compact) support in Σ, for which the infimum of the weighted (logarithmic) energy

$$I_\omega(\mu) \;=\; -\int\int \log|z - x|d\mu(z)d\mu(x) \;+\; 2\int \phi(x)d\mu(x)\,,\; \mu \in M(\Sigma)\,,$$

is attained, where, as usual, $M(\Sigma)$ denotes the collection of all positive unit Borel measures supported in Σ. Moreover, setting $F_\omega = I_\omega(\mu_\omega) - \int \phi d\mu_\omega$, we have the following property, which uniquely characterizes both the extremal measure μ_ω and its support $\operatorname{supp}\mu_\omega$:

$$V^{\mu_\omega}(z) + \phi(z) \begin{cases} \leq\; F_\omega\,, & \text{for } z \in \operatorname{supp}\mu_\omega\,, \\ \geq\; F_\omega\,, & \text{for quasi-every } z \in \Sigma\,, \end{cases}$$

where for a measure σ, $V^\sigma(z) = -\int \log|z - x|\,d\sigma(x)$, and a property is said to be satisfied for "quasi-every" z in a certain set, if it holds except in a possible subset of zero capacity (see e.g. [12]).

Indeed, let Σ be a closed set and ω an admissible weight on Σ. In [1], the authors considered asymptotically extremal polynomials $\{p_n\}$ with respect to varying weights $\{\omega_n\}_{n\in\mathbb{N}}$, that is, when it holds

$$\lim_{n\to\infty} \|\omega_n p_n\|^{1/n}_{L^\infty(\Sigma)} \;=\; \exp\left(-F_\omega\right)\,, \tag{10}$$

where $\omega_n^{1/n}$ tends to another weight ω in some sense, which will be indicated below. In particular, in [1, Th. 1.1] the authors proved the following result:

Theorem 1.1 *Let Σ be a closed interval in $\mathbb{R}$. If $\{\omega_n\}$ is a sequence of weights on Σ and ω is an admissible weight on Σ, in the sense of Definition 1.1, such that*

$$\liminf_{n\to\infty} \omega_n(x)^{1/n} \geq \omega(x)\,, \text{ for qu.e. } x \in \Sigma\,, \tag{11}$$

and $\{p_n\}$ a sequence of monic polynomials satisfying condition (10). Then, if we denote the zeros of p_n by $x_{n,k}$, $k = 1, \ldots, n$, it holds:

$$\frac{1}{n}\sum_{k=1}^{n} \delta_{x_{n,k}} \longrightarrow \mu_\omega\,, n \to \infty\,, \tag{12}$$

where $\delta_{x_{n,k}}$ denotes the Dirac Delta at the point $x_{n,k}$ and the convergence holds in the weak- topology.*

Finally, take into account that the equilibrium measure and the counterpart of Ullman's distribution (see [11, sect. 6]) for the interval $[0,1]$ are, respectively:

$$d\mu_{eq}(x) = \frac{1}{\pi}\frac{dx}{\sqrt{x(1-x)}}\,, \tag{13}$$

$$d\mu_\alpha(x) = \frac{\alpha}{\pi}\, x^{\alpha-1}\left(\int_x^1 \frac{t^{-\alpha-1/2}}{\sqrt{1-t}}\,dt\right)dx = \frac{\alpha}{\pi\sqrt{x}}\left(\int_x^1 \frac{t^{\alpha-1}}{\sqrt{t-x}}\,dt\right)dx\,. \tag{14}$$

Denote by S_n the n-th monic orthogonal polynomial with respect to the inner product (1) with weights ρ_k^2 given by (3)-(4). Now, the first main result deals with the weak zero asymptotics.

Theorem 1.2 *If we denote by $x_{n,j}^{(k)}$, $j = 1, \ldots, n-k$, the zeros of the k-th derivative $S_n^{(k)}$ of the polynomial S_n, $k = 0, 1, \ldots, M$, then, for $k \geq \overline{k}$:*

$$\frac{1}{n-k}\sum_{j=1}^{n-k} \delta_{x_{n,j}^{(k)}/a_{\overline{k},n}} \longrightarrow \begin{cases} d\mu_{eq}\,, & \text{if } \eta_{\overline{k}} > 0\,, \\ d\mu_{\alpha_{\overline{k}}}\,, & \text{if } \eta_{\overline{k}} = 0\,, \end{cases} \tag{15}$$

where the convergence holds in the weak- sense.*

Our second result is related to the n-root asymptotics. In order to establish it, we need a bound for the distance of the zeros to the convex hull of the support (in our case, the real semiaxis). Up to now, the unique sufficient condition to guarantee it is given in terms of certain hierarchy imposed to the weights in the Sobolev inner product. Following Durán and Saff in [2], suppose that we have a Sobolev inner product (1) with $d\mu_k(x) = w_k(x)dx, k = 0, \ldots, M$, supported on the real semiaxis $[0,\infty)$ and there exist positive constants $\{C_i\}_{i=0}^M$ such that the weights w_k satisfy that

$$C_k = \left\|\frac{w_k}{w_{k-1}}\right\|_{L^\infty(\mathbb{R}^+)}\,, k = 1, \ldots, M\,. \tag{16}$$

Theorem 1.3 *If the weights $w_k(x) = \rho_k^2(x), k = 0, \ldots, M$, satisfy condition (16), the monic k-th derivatives of the rescaled Sobolev orthogonal polynomials, $R_{n,k}(x) = \dfrac{(n-k)!}{n!}\, a_{0,n}^{-n}\, S_n^{(k)}(a_{0,n}x)$, $k = 0, \ldots, M$, satisfy the following asymptotic behavior, uniformly in compact subsets of $\mathbb{C}\setminus\mathbb{R}^+$:*

$$\lim_{n\to\infty} |R_{n,k}(x)|^{1/n} = \begin{cases} \frac{1}{2}\left|x-\frac{1}{2}+\sqrt{x(x-1)}\right|\,, & \textit{if}\, \eta_0>0\,,\\ \frac{1}{2e^{1/\alpha_0}}\left|x-\frac{1}{2}+\sqrt{x(x-1)}\right| e^{\zeta_{\alpha_0}(x)}\,, & \textit{if}\, \eta_0=0\,, \end{cases} \tag{17}$$

where $\zeta_{\alpha_0}(x) = \Re\int_0^1 \frac{x t^{\alpha_0-1}}{\sqrt{x(x-t)}}dt\,.$

The rest of the paper is organized as follows. In section 2 we prove Theorem 1.2. In Section 3, a simpler proof of the result in [2] about an upper bound for the distance of the zeros to the convex hull of the support and, as a consequence, the proof of Theorem 1.3 are given.

2 Weak zero asymptotics

The goal of this section is to prove Theorem 1.2 about the weak asymptotics for (rescaled) Sobolev polynomials with exponential weights in the real semiaxis. Having in mind the result in Theorem 1.1, we will show that these polynomials are asymptotically extremal. For it, it plays a key role the asymptotic behavior of the related standard orthogonal polynomials.

First, let us remind some basic facts concerning exponential weights.

Lemma 2.1 *(a) Markov and Nikolskii type inequalities: For any polynomial P of degree $\leq n$,*

$$\|P'\rho_k\|_{L^p(\mathbb{R}^+)} \leq A_n\,\|P\rho_k\|_{L^p(\mathbb{R}^+)}\,, \tag{18}$$

where A_n is such that $A_n^{1/n} \underset{n}{\longrightarrow} 1$ and $0<p\leq\infty$, and

$$B_n\,\|P\,\rho_k\|_{L^\infty(\mathbb{R}^+)} \leq \|P\,\rho_k\|_{L^2(\mathbb{R}^+)} \leq C_n\,\|P\,\rho_k\|_{L^\infty(\mathbb{R}^+)}\,, \tag{19}$$

where the constants B_n and C_n are such that $B_n^{1/n} \underset{n}{\longrightarrow} 1$ and $C_n^{1/n} \underset{n}{\longrightarrow} 1$.

(b) Norm asymptotics: Let us denote by $L_{k,n}$ the n-th monic orthogonal polynomial with respect to the weight ρ_k^2 given by (3)-(4)*, then:*

$$\lim_{n\to\infty} a_{k,n}^{-1}\,\|L_{k,n}\rho_k\|_{L^\infty[0,1]}^{1/n} = \begin{cases} \frac{1}{4}\,, & \textit{if}\;\; \eta_k>0\\ \frac{1}{4}\,e^{-1/\alpha_k}\,, & \textit{if}\;\; \eta_k=0\,. \end{cases} \tag{20}$$

Proof:

Markov inequality (18) is given in [7, Theorem 1.6] and the proof of Nikolskii type inequality (19) is similar as that given in [5, Theorem 10.3].

The proof of the norm asymptotics (20) may be found in [11, sect. 6] for the Freud case ($\eta_k=0$). For the Erdös case ($\eta_k>0$), take into account that to any weight $g(x)$ supported on $[0,\infty)$ we can associate the weight f supported on $\mathbb{R}$, given by $f(x)=|x|g(x^2)$, in such a way that if p_n and q_n denote the monic orthogonal polynomials with respect to f and g, respectively, we have that $\|q_n g\|_{L^2[0,\infty)} = \|p_{2n} f\|_{L^2(\mathbb{R})}$ (see [11]). Thus, it suffices to use the results in [4]. □

In addition, for the proof of Theorems 1.2 and 1.3 we need to calculate the potentials of measures (13) and (14). The results are summarized in the following

Lemma 2.2 *The potentials of measures* (13) *and* (14) *for* $z \notin [0,1]$ *are given, respectively, by:*

$$V^{\mu_{eq}}(z) \;=\; \log 2 - \log\left| z - \frac{1}{2} + \sqrt{z(z-1)}\right| \,, \tag{21}$$

$$V^{\mu_{\alpha}}(z) \;=\; \log 2 + \frac{1}{\alpha} - \log\left| z - \frac{1}{2} + \sqrt{z(z-1)}\right| - \zeta_{\alpha}(z)\,, \tag{22}$$

where $\zeta_{\alpha}(x) = \Re \displaystyle\int_0^1 \frac{x t^{\alpha-1}}{\sqrt{x(x-t)}}\, dt$. *Moreover, for the weight* $\omega \equiv 1$ *on* $[0,1]$, *we have that* $\mu_{\omega} = \mu_{eq}$ *and* $F_{\omega} = 2\log 2$. *For the weight* $\omega(z) = \exp\left(-\gamma_{\alpha} z^{\alpha}\right)$ *on* $[0,1]$, *the corresponding equilibrium measure is given by* μ_{α} (14). *In this case,* $F_{\omega} = 2\log 2 + \frac{1}{\alpha}$.

Proof: (21) is immediate, since it is the well-known expression for the potential of the equilibrium measure in a compact interval of $\mathbb{R}$ (see e.g. [5, pp. 36-37]). Thus, taking $\phi = log\omega \equiv 0$ in $[0,1]$, we have that for $z \in [0,1]\,,\, V^{\mu_{eq}}(z) + \phi(z) = 2\log 2 = F_{\omega}\,.$

For the proof of (22), we have from :

$$V^{\mu_{\alpha}}(z) \;=\; \frac{\alpha}{\pi}\int_0^1 \log\frac{1}{|z-x|}\left\{\int_x^1 \frac{t^{\alpha-1}}{\sqrt{x(t-x)}}\,dt\right\}\,dx \;=\; \int_0^1 \alpha t^{\alpha-1}\left\{\frac{1}{\pi}\int_0^t \frac{\log\frac{1}{|z-x|}}{\sqrt{x(t-x)}}\,dx\right\}\,dt.$$

The last integral above represents the potential of the equilibrium measure for the interval $[0,t]$. Thus, again from [5, pp. 36-37], we have:

$$\frac{1}{\pi}\int_0^t \frac{\log\frac{1}{|z-x|}}{\sqrt{x(t-x)}}\,dx \;=\; \begin{cases} \log\left(\frac{4}{t}\right)\,, & z \in [0,t]\,, \\ \log 2 - \log\left|z - \frac{t}{2} + \sqrt{z(z-t)}\right|\,, & z \notin [0,t]\,, \end{cases}$$

and so, it yields for $z \notin [0,1]$,

$$\begin{aligned} V^{\mu_{\alpha}}(z) \;&=\; \int_0^1 \alpha t^{\alpha-1}\left(\log 2 - \log\left|z - \frac{t}{2} + \sqrt{z(z-t)}\right|\right)\,dt \;= \\ &\log 2 - \int_0^1 \alpha t^{\alpha-1}\log\left|z - \frac{t}{2} + \sqrt{z(z-t)}\right|\,dt\,, \end{aligned}$$

and for $z \in [0,1]$,

$$\begin{aligned} V^{\mu_{\alpha}}(z) \;&=\; \int_0^z \alpha t^{\alpha-1}\left(\log 2 - \log\left|z - \frac{t}{2} + \sqrt{z(z-t)}\right|\right)\,dt \;+\; \int_z^1 \alpha t^{\alpha-1}\log\left(\frac{4}{t}\right)\,dt \;= \\ &\log 2\int_0^1 \alpha t^{\alpha-1}dt + \int_z^1 \alpha t^{\alpha-1}\log\left(\frac{2}{t}\right)\,dt \;-\; \int_0^z \alpha t^{\alpha-1}\log\left|z - \frac{t}{2} + \sqrt{z(z-t)}\right|\,dt \;= \\ &\log 2\,(2 - z^{\alpha}) + z^{\alpha}\log z + \frac{1 - z^{\alpha}}{\alpha} \;-\; \int_0^z \alpha t^{\alpha-1}\log\left|z - \frac{t}{2} + \sqrt{z(z-t)}\right|\,dt\,. \end{aligned}$$

Now, making the change $t = zu$ and integrating by parts, we have:

$$\int_0^z \alpha t^{\alpha-1} \log \left| z - \frac{t}{2} + \sqrt{z(z-t)} \right| dt \;=\; z^\alpha \left(\log z + \int_0^1 \alpha u^{\alpha-1} \log \left(1 - \frac{u}{2} + \sqrt{1-u} \right) \right) \;=$$
$$z^\alpha \left(\log z - \log 2 + \frac{1}{2} \int_0^1 u^\alpha \frac{1+\sqrt{1-u}}{\left(1 - \frac{u}{2} + \sqrt{1-u}\right)\sqrt{1-u}}\, du \right) \;=$$
$$z^\alpha \left(\log z - \log 2 + \int_0^1 \left(\frac{1}{\sqrt{1-u}} - 1 \right) du \right) \;=\; z^\alpha \left(\log z - \log 2 + \gamma_\alpha - \frac{1}{\alpha} \right).$$

Thus, we obtain for $z \in [0,1]$:

$$V^{\mu_\alpha}(z) \;=\; 2\log 2 + \frac{1}{\alpha} - \gamma_\alpha z^\alpha \,. \tag{23}$$

Similarly, we have for $z \in \mathbb{R} \setminus [0,1]$:

$$V^{\mu_\alpha}(z) \;=\; \log 2 - \log \left| z - \frac{1}{2} + \sqrt{z(z-1)} \right| \;-\; \int_0^1 \frac{z t^{\alpha-1}}{\sqrt{z(z-t)}}\, dt \;+\; \frac{1}{\alpha}\,, \tag{24}$$

and the conclusion is the same for $z \in \mathbb{C} \setminus \mathbb{R}$, but taking real part in the integral in (24).

From (23), taking $\phi(z) = -\log \omega(z) = \gamma_\alpha z^\alpha$, we have that when $z \in [0,1]$:

$$V^{\mu_\alpha}(z) + \phi(z) = 2\log 2 + \frac{1}{\alpha} = F_\omega\,,$$

and, thus, it is clear that $\mu_\omega = \mu_\alpha$. This settles the proof. □

Proof of Theorem 1.2

Taking into account the extremality of $L_{k,n}$ and S_n with respect to $\|\cdot\|_{L^2(\rho_k^2)}$ and the Sobolev norm $\|\cdot\|_S$ (2), respectively, the Markov inequalities (18) and (9), we can write, using the fact that $\rho_{\overline{k}}$ is the main weight,

$$\frac{n!}{(n-\overline{k})!} \|L_{\overline{k},n-\overline{k}}\, \rho_{\overline{k}}\|_{L^2(\mathbb{R}^+)} \le \|S_n^{(\overline{k})} \rho_{\overline{k}}\|_{L^2(\mathbb{R}^+)} \le \|S_n\|_S \le \|L_{\overline{k},n}\|_S \le C(n) \|L_{\overline{k},n}\, \rho_{\overline{k}}\|_{L^2(\mathbb{R}^+)}\,, \tag{25}$$

with $\lim\limits_{n\to\infty} C(n)^{1/n} = 1$. So, taking into account that $\overline{k}$ is a fixed nonnegative integer and $\lim\limits_{n\to\infty} \left(\frac{n!}{(n-\overline{k})!} \right)^{1/n} = 1$, (20) and (25) yield:

$$\lim_{n\to\infty} a_{\overline{k},n}^{-1} \|S_n\|_S^{1/n} \;=\; \begin{cases} \frac{1}{4}\,, & \text{if } \eta_{\overline{k}} > 0\,, \\ \frac{1}{4}\, e^{-1/\alpha_{\overline{k}}}\,, & \text{if } \eta_{\overline{k}} = 0\,. \end{cases}$$

and

$$\lim_{n\to\infty} a_{\overline{k},n}^{-1} \|S_n^{(\overline{k})}\, \rho_{\overline{k}}\|_{L^2(\mathbb{R}^+)}^{1/n} \;=\; \begin{cases} \frac{1}{4}\,, & \text{if } \eta_{\overline{k}} > 0\,, \\ \frac{1}{4}\, e^{-1/\alpha_{\overline{k}}}\,, & \text{if } \eta_{\overline{k}} = 0\,. \end{cases}$$

Now, applying the Nikolskii inequalities for exponential weights (19), we have

$$\lim_{n\to\infty} a_{\overline{k},n}^{-1}\,\|S_n^{(\overline{k})}\rho_{\overline{k}}\|_{L^\infty(\mathbb{R}^+)}^{1/n} \;=\; \begin{cases} \frac{1}{4}\,, & \text{if } \eta_{\overline{k}} > 0\,,\\ \frac{1}{4}\,e^{-1/\alpha_{\overline{k}}}\,, & \text{if } \eta_{\overline{k}} = 0\,. \end{cases} \tag{26}$$

Thus, rescaling the polynomial S_n, that is, setting $R_n(x) \;=\; a_{\overline{k},n}^{-n} S_n(a_{\overline{k},n}x)$, taking $\omega_{\overline{k},n}(x) \;=\; \rho_{\overline{k}}(a_{\overline{k},n}x)$ and applying (6), (26) yields

$$\lim_{n\to\infty} \|R_n^{(\overline{k})}\omega_{\overline{k},n}\|_{L^\infty[0,1]}^{1/n} \;=\; \begin{cases} \frac{1}{4}\,, & \text{if } \eta_{\overline{k}} > 0\,,\\ \frac{1}{4}\,e^{-1/\alpha_{\overline{k}}}\,, & \text{if } \eta_{\overline{k}} = 0\,, \end{cases} \tag{27}$$

and thus, by the Markov inequalities (18), we have that (27) also holds replacing $R_n^{(\overline{k})}$ by any derivative $R_n^{(k)}$, with $k \geq \overline{k}$.

Observe that if $\eta_{\overline{k}} > 0$ (that is when the main weight is of Erdös type), we have that $\lim\limits_{n\to\infty} \sqrt[n]{\omega_{\overline{k},n}(x)} = 1\,,$ qu.e. $x \in [0,1]\,,$ (in fact, the convergence does not hold only at the endpoints). Therefore, applying Theorem 1.1, we conclude that the contracted zeros of $R_n^{(k)}$, with $k \geq \overline{k}$, asymptotically follow μ_{eq} (13). If $\eta_k = 0$, i.e. the main weight is of Freud type, we have that $\sqrt[n]{\omega_{\overline{k},n}(x)} \;=\; \exp\left(-\gamma_{\alpha_{\overline{k}}} x^{\alpha_{\overline{k}}}\right)\,,$ for each $n \in \mathbb{N}$, and so, Theorem 1.1 and Lemma 2.2 show that the zeros of $R_n^{(k)}$, $k \geq \overline{k}$, distribute according to the measure $\mu_{\alpha_{\overline{k}}}$, given by (14). $\square$

3 Logarithmic asymptotics

In order to prove the result on logarithmic asymptotics of Sobolev orthogonal polynomials, the boundness of the distance of their zeros to real semiaxis must be established. This is announced in the following lemma, which is related to a result given by Durán and Saff in [2], within a more general framework. However, for the sake of completeness, a simpler proof will be given.

Lemma 3.1 *Suppose that we have a Sobolev inner product* (1) *with* $d\mu_k(x) = w_k(x)\,dx, k = 0,\ldots,M$, *supported on a subset* I *of the real axis. Assume that there exist positive constants* $\{C_i\}_{i=0}^M$ *such that the weights* w_k *satisfy the condition:*

$$C_k \;=\; \left\|\frac{w_k}{w_{k-1}}\right\|_{L^\infty(I)}\,,\; k = 1,\ldots,M\,. \tag{28}$$

Then, there exists another positive constant C, *only depending on* $\{C_i\}_{i=0}^M$, *such that if* z_0 *is a zero of the Sobolev polynomial* S_n, *we have:*

$$d(z_0, \mathrm{Co}(I)) \leq C\,,\;\; \textit{with}\; C = \frac{1}{2}\sqrt{\sum_{k=1}^{M} k^2 C_k}\,, \tag{29}$$

where $\mathrm{Co}(I)$ *denotes the convex hull of the set* I *and* $d(z_0, \mathrm{Co}(I))$ *is the distance between* z_0 *and* $\mathrm{Co}(I)$.

To prove Lemma 3.1 we need a technical result (see [1, Lemma 3.4]):

Lemma 3.2 *If* $a_k > 0$, $k = 0, \ldots, m$, *then*

$$\sum_{k=1}^{m} a_{k-1} a_k \leq \frac{1}{4} \left(\sum_{k=0}^{m} a_k \right)^2$$

Now, we are concerned with the proof of Lemma 3.1. Indeed, let $z_0 = x_0 + iy_0$ be a zero of a n-th Sobolev orthogonal polynomial S_n with respect to the inner product:

$$\langle p, q \rangle_S := \sum_{i=0}^{M} \int p^{(i)}(x)\, \overline{q}^{(i)}(x)\, w_i(x) dx, \quad M \in \mathbb{Z}_+ ,$$

Thus, $S_n(z) = (z - z_0) q(z)$, for some polynomial q of degree $n-1$. Then, we have that $\langle (z - z_0) q(z), q(z) \rangle_S = 0$, and so,

$$\sum_{k=0}^{M} \int (x - z_0) \left| q^{(k)}(x) \right|^2 w_k(x) dx + \sum_{k=1}^{M} k \int q^{(k-1)}(x) \overline{q}^{(k)}(x) w_k(x) dx = 0$$

Therefore, making use of the mean value theorem for integrals, we have for some $\tau \in \mathrm{Co}(I)$:

$$\begin{aligned} (\tau - z_0) \int \textstyle\sum_{k=0}^{M} \left| q^{(k)}(x) \right|^2 w_k(x) dx &= \int (x - z_0) \textstyle\sum_{k=0}^{M} \left| q^{(k)}(x) \right|^2 w_k(x) dx \\ &= -\textstyle\sum_{k=1}^{M} k \int q^{(k-1)}(x) \overline{q}^{(k)}(x) w_k(x) dx \end{aligned}$$

and so,

$$|\tau - z_0| \|q\|_S = \left| \sum_{k=1}^{M} k \int q^{(k-1)}(x) \overline{q}^{(k)}(x) w_k(x) dx \right| .$$

Therefore:

$$d(z_0, \mathrm{Co}(I)) \|q\|_S \leq |\tau - z_0| \|q\|_S = \left| \sum_{k=1}^{M} k \int q^{(k-1)}(x) \overline{q}^{(k)}(x) w_k(x) dx \right| ,$$

and thus, by applying the Cauchy-Schwarz inequality, both for integrals and for summatories, condition (28) and Lemma 3.2, we get:

$$d(z_0, \mathrm{Co}(I)) \|q\|_S^2 \leq \frac{1}{2} \sqrt{\sum_{k=1}^{M} k^2 C_k} \;\; \|q\|_S^2 ,$$

and hence,

$$d(z_0, \mathrm{Co}(I)) \leq \frac{1}{2} \sqrt{\sum_{k=1}^{M} k^2 C_k}$$

and this completes the proof.

□

Proof of Theorem 1.3

Take into account that if the unit zero counting measures of a sequence of polynomials $\{p_n\}$ converge (in the weak-* topology) to a certain measure μ with compact support, and K is a compact subset of $\mathbb{C} \setminus \mathrm{Co}(\operatorname{supp} \mu)$ without limit points of zeros of $\{p_n\}$, then:

$$\lim_{n\to\infty} |p_n(x)|^{1/n} = \exp(-V^{\mu}(x)), \tag{30}$$

uniformly on K. On the other hand, Lemma 3.1 yields that for each n, the zeros of the monic rescaled Sobolev orthogonal polynomial $R_{n,0}(x) = R_n(x) = a_{0,n}^{-n} S_n(a_{0,n}x)$ lie in the half-strip

$$\left\{ z \in \mathbb{C} / -\frac{C}{a_{0,n}} \leq \Im(z) \leq \frac{C}{a_{0,n}}, \Re(z) \geq -\frac{C}{a_{0,n}} \right\},$$

with the constant C given by (29). Since Theorem 1.2 implies that these zeros asymptotically distribute as μ_{eq}, when $\eta_k > 0$, and as μ_{α_0}, when $\eta_k = 0$, then taking into account (30), the expression of their potentials (21)-(22) and the asymptotic behavior of $a_{0,n}$ (see (7)-(8)) yield (17). $\square$

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